Vector Calculus 20E, Winter 2016, Lecture A, Final Exam

Three hours, eight problems. No calculators allowed.

Please start each problem on a new page.

You will get full credit only if you show all your work clearly.

Simplify answers if you can, but don't worry if you can't!

1. Let R be the solid region between the spheres of radius 1 and 2, centred on the origin. Compute

$$\int_R (x+y)^2 \; \mathrm{d}V.$$

2. Let Σ be the surface which is given by $z = 1 - x^2 - y^2, z \geq 0$. Compute the surface area of Σ.

3. Let Σ be the part of the unit sphere $x^2+y^2+z^2 = 1$ which lies above the plane $z = \frac{1}{2}$. Compute the average of the function $f(x,y,z) = z$ over Σ.

4. Let γ be the oriented curve parametrised by $t \mapsto (t, t^2, t^3)$ for $0 \leq t \leq 1$ and let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x,y,z) = (x+z, y^3, 1-x)$. Compute

$$\int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

5. Let R be the region $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$, and let Σ be its boundary surface, oriented outwards. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x,y,z) = (4x - z^2, x + 3z, 6 - z)$. Compute

$$\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

6. Let Σ be the part of the cone given by $z = \sqrt{x^2+y^2}, 0 \leq z \leq 1$, oriented outwards. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x,y,z) = (-z^2 y, z^2 x, z^4)$. Compute

$$\int_\Sigma (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}.$$

7. Let Σ be the unit hemisphere $x^2+y^2+z^2 = 1, x \geq 0$, oriented using the outward normal, and let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x,y,z) = (y, x, z)$. Compute the flux

$$\int_\Sigma \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

8. Find the potential function ϕ which satisfies $\phi(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0$ and $\nabla\phi = \mathbf{F}$, where $\mathbf{F}$ is the vector field given by $\mathbf{F}(x,y,z) = (\sin y - z\cos x, x\cos y + \sin z, y\cos z - \sin x)$.

Vector Calculus 20E, Winter 2017, Lecture B, Final Exam

Three hours, eight problems. No calculators allowed.

Please start each problem on a new page.

You will get full credit only if you show all your work clearly.

Simplify answers if you can, but don't worry if you can't!

1. Let Σ be the part of the plane $2x + y + z = 6$ where $x, y, z \geq 0$. Compute

$$\int_{\Sigma} (x + z)\ \mathrm{d}A.$$

2. Let Σ be the piece of a cylinder given by $x^2 + y^2 = 4$, $y \geq 0$ and $0 \leq z \leq 1$, oriented with the outward normal. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (x, 1, z^2)$. Compute

$$\int_{\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

3. Let Σ be the hemisphere given by $(x-1)^2 + y^2 + z^2 = 1$ and $z \geq 0$, oriented with the upward normal. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (-e^{\sin z} y, x + \sin z, \cos x)$. Compute

$$\int_{\Sigma} (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{A}.$$

4. Let R be the solid pipe defined by $1 \leq x^2 + y^2 \leq 4$ and $0 \leq z \leq 10$, and let Σ be its boundary surface, oriented out of R. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (x + yz, y + x^2 z^3, xyz)$. Compute

$$\int_{\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{A}.$$

5. Let R be the part of the unit ball given by $x^2 + y^2 + z^2 \leq 1$, $x, y, z \geq 0$. Find the average, over R, of the distance from the origin.

6. Let γ be some curve which runs from $(0, 0, 0)$ to $(1, 2, 3)$. Let $\mathbf{F}$ and $\mathbf{G}$ be vector fields given by $\mathbf{F}(x, y, z) = (3x^2y^2z, 2x^3yz, x^3y^2)$ and $\mathbf{G}(x, y, z) = (3x^2y^2z, 2x^3yz, x^2y^3$. Evaluate, or say why it can't be evaluated without further information, the integrals

$$\int_{\gamma} \mathbf{F} \cdot \mathrm{d}\mathbf{s} \qquad \int_{\gamma} \mathbf{G} \cdot \mathrm{d}\mathbf{s}.$$

7. Let γ be the straight-line segment running from $(1, 2, 3)$ to $(4, 0, 2)$. Let $\mathbf{F}$ be the vector field given by $\mathbf{F}(x, y, z) = (xz, 3y, 1)$. Calculate

$$\int_{\gamma} \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

8. Let D be the parallelogram whose vertices are $(0, 1), (1, 0), (2, 2), (3, 1)$. Evaluate

$$\int_{D} \cos(x + y)\ \mathrm{d}A.$$